Existence of a maximal solution of singular parabolic equations with absorptions: quenching phenomenon and the instantaneous shrinking phenomenon

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Abstract. This paper deals with nonnegative solutions of the one dimensional degenerate parabolic equations with zero homogeneous Dirichlet boundary condition. To obtain an existence result, we prove a sharp gradient estimate of $|u_x|$. Besides, we investigate the behaviors of nonnegative solutions such as the quenching phenomenon, and the finite speed of propagation. Our results of the Dirichlet problem will be extended to the associated Cauchy problem. In addition, we show that the phenomenon of the instantaneous shrinking of compact support of the nonnegative solutions occurs if f satisfies some growth condition.

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1 Introduction

In this paper, we study the nonnegative solutions of the one dimensional degenerate parabolic equation on a given open bounded interval $I = (-l, l)$

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } I \times (0,\infty), \\ u(-l,t) = u(l,t) = 0 & t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{1}$$

where $\beta \in (0,1)$, $p > 2$; and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$, i.e

$$\chi_{\{u>0\}} = \begin{cases} 1, & \text{if } u > 0, \\ 0, & \text{if } u \leq 0. \end{cases}$$

Note that the absorption term $\chi_{\{u>0\}}u^{-\beta}$ becomes singular when u is near to 0, and we impose $\chi_{\{u>0\}}u^{-\beta} = 0$ whenever $u = 0$. Through this paper, we assume that $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}(\mathbb{R})$ is a nonnegative function. But, f will be addressed in detail later for the existence of solution, see (H_1) and (H_2) below.

As already known, problem (1) in the semi-linear case ($p = 2$, and $f = 0$) can be considered as a limit of mathematical models arising in Chemical Engineering corresponding to catalyst kinetics of Langmuir-Hinshelwood type (see, e.g. [27] p. 68, [24] and reference therein). The semi-linear case was studied in many papers such as [24], [18], [22], [9], [7], [28], and so forth. These papers focused on studying the existence of solution, and the behaviors of solutions. From our knowledge, the existence result of the semi-linear case was first proved by Phillips for the Cauchy problem (see Theorem 1, [24]). The same result holds for the semi-linear equation with positive Dirichlet boundary condition (see Theorem 2, [24]). Moreover, he proved a property of the finite speed of propagation of nonnegative solutions, i.e, any solution with compact support initially has compact support at all later times $t > 0$.

The semi-linear problem of this type was also extended in many aspects. In [9], J. Davila, and M. Montenegro proved the existence of solution with zero Dirichlet boundary condition with a source term $f(u)$. We emphasize that the equations of this type with zero Dirichlet boundary condition are harder than the one of positive Dirichlet boundary condition because of the effect of the singular term $u^{-\beta}\chi_{\{u>0\}}$. Furthermore, they showed that the uniqueness result holds for a particular class of positive solutions, see Theorem 1.10 in [9]. Recently, Diaz et al., [7], proved a uniqueness result for a class of solutions, which is different from the one of [9]. However, Winkler showed that the uniqueness result fails in general (see Theorem 1.1, [28]).

After that, the equations of this type was considered under more general forms. For example, the case of quasilinear diffusion operators was already considered in [18] (for a different diffusion term). We also mention here the porous medium of this type was studied by B. Kawohl and R. Kersner, [19]. We note that problem (1) was considered recently by Giacomoni et al., [15] with $f(u)$ on the right hand side, but there was a technical fault in the proof of the existence of solution.

Inspired by the above studies, we would like to investigate the existence of nonnegative solutions and the behaviors of solutions of equation (1). Before stating our main results, let us define the notion of a weak solution of equation (1).

Definition 1 *Given $0 \leq u_0 \in L^1(I)$. A function $u \geq 0$ is called a weak solution of equation* (1) *if $f(u), u^{-\beta}\chi_{\{u>0\}} \in L^1(I\times(0,\infty))$, and $u \in L^p_{loc}(0,\infty; W^{1,p}_0(I)) \cap L^\infty_{loc}(\overline{I}\times(0,\infty)) \cap \mathcal{C}([0,\infty); L^1(I))$ satisfies equation* (1) *in the sense of distributions $\mathcal{D}'(I \times (0,\infty))$, i.e,*

$$\int_0^\infty \int_I -u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi \, dxdt = 0, \quad \forall \phi \in \mathcal{C}^\infty_c(I \times (0,\infty)). \tag{2}$$

Next, if f satisfies either (H_1) or (H_2) below, we have then an existence of solution of problem (1).

(H_1) $\quad f \in \mathcal{C}^1(\mathbb{R})$ and $f(0) = 0$.

(H_2) $\quad f$ is a nondecreasing function, and $f(0) = 0$.

Theorem 2 *Let $0 \leq u_0 \in L^\infty(I)$, and f satisfy (H_1). Then, there exists a maximal weak bounded solution u of equation* (1). *Moreover, we have*

There exists a positive constants $C(\beta, p)$ such that

$$|u_x(x,t)| \leq C.u^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}\|u_0\|_\infty^{\frac{1+\beta}{p}} + M_f(u_0).\|u_0\|_\infty^{\frac{\beta}{p}} + M_{f'}(u_0).\|u_0\|_\infty^{\frac{1+\beta}{p}} + 1\right), \tag{3}$$

for a.e $(x,t) \in I \times (0,\infty)$, with $\gamma = \dfrac{p}{p+\beta-1}$, and $M_g(u_0) = \left(\max\limits_{0\leq s\leq 2\|u_0\|_\infty} |g(s)|\right)^{\frac{1}{p}}$, for any $g \in \mathcal{C}(\mathbb{R})$.

As a consequence of (3), *for any $\tau > 0$ there is a positive constant $C(\beta, p, \tau, \|u_0\|_\infty)$ such that*

$$|u(x,t) - u(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{2}}\right), \quad \forall x, y \in \overline{I}, \quad \forall t, s \geq \tau. \tag{4}$$

Theorem 3 *Let $0 \leq u_0 \in L^1(I)$, and f satisfy (H_2). Then, there exists a maximal weak solution u of equation* (1). *Furthermore, we have*

For any $\tau > 0$, there exist two positive constants $C_1(\beta, p, |I|)$ and $C_2(p, |I|)$ such that

$$|u_x(x,t)| \leq C_1.u^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + \tau^{-\frac{\beta}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{\beta}{\lambda}}.m_f(\tau, u_0) + 1\right), \tag{5}$$

for a.e $(x,t) \in I \times (\tau,\infty)$, with $\lambda = 2(p-1)$, and $m_f(\tau, u_0) = f^{\frac{1}{p}}\left(C_2.\tau^{-\frac{1}{\lambda}}\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}\right)$.

As a consequence of (5), *there is a positive constant $C(\beta, p, \tau, |I|, \|u_0\|_{L^1(I)})$ such that*

$$|u(x,t) - u(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{2}}\right), \quad \forall x, y \in \overline{I}, \quad \forall t, s \geq \tau. \tag{6}$$

Remark 4 *Note that estimate* (5) *does not include the term of* f'*, compare with* (3)*. Actually, this one is a combination of estimate* (3) *without* $M_{f'}(u_0)$*, and the smoothing effect* $L^1 - L^\infty$*.*

Remark 5 *Conclusion* (4) *(resp.* (6)*) implies that* u *is continuous up to the boundary. This result answers an open question stated in the Introduction of [28] for the semi-linear case.*

Remark 6 *When* $p = 2$ *and* $f = 0$*, estimate* (3) *becomes the gradient estimates in [24], [9], [28].*

Remark 7 *The condition* $f(0) = 0$ *in* (H_1) *and* (H_2) *is necessary for the existence of nonnegative solutions. If* f *violates this one, i.e,* $f(0) > 0$ *then the existence result fails, see Corollary 30.*

A second goal of this article is to study the most striking phenomenon of equations of this type, the so called quenching phenomenon that solution vanishes after a finite time. This property arises due to the presence of the singular term $u^{-\beta}\chi_{\{u>0\}}$. It occurs even starting with a positive unbounded initial data and there is a lack of uniqueness of solutions (see Theorem 1.1, [28] again). Then we have the following results

Theorem 8 *Assume as in Theorem* 2*. Let* v *be any weak solution of equation* (1)*. Then, there is a finite time* $T_0 = T_0(\beta, p, \|u_0\|_\infty)$ *such that*

$$v(t) = 0, \quad \text{for } t \geq T_0.$$

Theorem 9 *Assume as in Theorem* 3*. Let* v *be any weak solution of equation* (1)*. Then, there is a finite time* $T_0 = T_0(\beta, p, |I|, \|u_0\|_{L^1(I)})$ *such that*

$$v(t) = 0, \quad \text{for } t \geq T_0.$$

Besides, we shall investigate the existence of solution of the Cauchy problem associated to equation (1).

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0, & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}. \end{cases} \tag{7}$$

Moreover, we also study behaviors of solutions of Cauchy problem such as the quenching phenomenon, and the finite speed of propagation. In particular, we show that if f satisfies a certain growth condition at infinity, then any weak solution has the instantaneous shrinking of compact support (in short ISS), namely, if u_0 only goes to 0 uniformly as $|x| \to \infty$, then the support of any weak solution is bounded for any $t > 0$. Concerning the ISS phenomenon, we refer to [6], [13], [16], and reference therein. Then, our main result of the Cauchy problem is as follows

Theorem 10 *Let* $0 \leq u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*. Assume that* f *satisfies either* (H_1) *or* (H_2)*. Then, there exists a weak bounded solution* $u \in \mathcal{C}([0, \infty); L^1(\mathbb{R})) \cap L^p(0, T; W^{1,p}(\mathbb{R}))$*, satisfying equation* (7) *in* $\mathcal{D}'(\mathbb{R} \times (0, \infty))$*.*

i) Furthermore, any solution with compact support initially has compact support for any $t > 0$*. And, the solution* u *constructed above is a maximal solution of equation* (7)*.*

ii) In addition, if $u_0(x) \to 0$ uniformly as $x \to \infty$, and f satisfies the following growth condition at infinity:

$$(H_3) \quad \textit{There is a real number } q_0 \in (0,1) \textit{ such that } f(s) \geq s^{q_0}, \textit{ when } s \to +\infty,$$

then such a weak solution of problem (7) *has ISS property.*

Remark 11 *We note that our results above also hold for the case where f is only a global Lipschitz function with $f(0) = 0$, see Remark 20, Remark 40, and Theorem 25.*

The paper is organized as follows: Section 2 is devoted to prove a sharp gradient estimate, which is the main key of proving the existence of solution. In section 3, we shall give the proof of Theorem 3, and Theorem 2 is proved in the same way. Section 4 is devoted to study the quenching phenomenon (including the proofs of Theorem 9 and Theorem 8). Finally, Section 5 concerns studying the existence of solution of the associated Cauchy problem, and behaviors of solutions, thereby includes the proof of Theorem 10.

Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(p, \beta, \tau)$ means that C only depends on p, β, τ. We also denote by $I_r(x) = (x - r, x + r)$ to the open ball with center at x and radius $r > 0$ in $\mathbb{R}$. If $x = 0$, we denote $I_r(0) = I_r$. Next $\partial_x u$ (resp. $\partial_t u$) means the partial derivative with respect to x (resp. t). We also write $\partial_x u = u_x$. Finally, the L^∞-norm of u is denoted by $\|u\|_\infty$.

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2 A sharp gradient estimate

In this part, we shall modify Bernstein's technique to obtain estimates on $|u_x|$, so called the gradient estimate in N-dimension. Roughly speaking, the gradient estimates that we shall prove are of the type

$$|u_x(x,t)| \leq C_1.u^{1-\frac{1}{\gamma}}(x,t)\left(1 + C_2(f, f')\right), \quad \text{for a.e } (x,t) \in I \times (0, \infty), \tag{8}$$

where the constant C_1 merely depends on the parameters β, p, while C_2 involves the terms of f and f'. It is well known that such a gradient estimate of (8) plays a crucial role in proving the existence of solution (see, e.g. [24], [9], [28] for the semi-linear case; and see [19] for the porous medium of this type). The degeneracy of the diffusion operator as $p > 2$ leads, obviously, to a considerable amount of additional technical difficulties. In the case $f = 0$, it is not difficult to show that estimate (8) becomes an equality for a suitable constant C_1 ($C_2 = 0$), when considering the stationary equation of (1). That is the reason why such a gradient estimate of this type is called *a sharp gradient estimate* (since the power of u in (8) cannot bigger or smaller than $1 - 1/\gamma$). By the appearance of the nonlinear diffusion, p-laplacian, we shall establish previously the gradient estimates for the solutions of the following regularizing problem.

For any $\varepsilon > 0$, let us set

$$g_\varepsilon(s) = s^{-\beta}\psi_\varepsilon(s), \text{ with } \psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon}),$$

and $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \le \psi \le 1$ is a non-decreasing function such that $\psi(s) = \begin{cases} 0, & \text{if } s \le 1, \\ 1, & \text{if } s \ge 2. \end{cases}$
Now fix $\varepsilon > 0$, we consider the following problem

$$(P_{\varepsilon,\eta}) \begin{cases} \partial_t z - (a(z_x)z_x)_x + g_\varepsilon(z) + f(z)\psi_\varepsilon(z) = 0, & \text{in } I \times (0,\infty), \\ z(-l,t) = z(l,t) = \eta, & t \in (0,\infty), \\ z(x,0) = z_0(x) + \eta, & x \in I, \end{cases} \tag{9}$$

with $a(s) = b(s)^{\frac{p-2}{2}}$, $b(s) = |s|^2 + \eta^\alpha$; $\alpha > 0$ will be addressed later; and $\eta \to 0^+$. Note that $a(z_x)$ is a regularization of $|z_x|^{p-2}$. Then, problem $(P_{\varepsilon,\eta})$ can be understood as a regularization of equation (1). The gradient estimates, presented in this framework are as follows:

Lemma 13 *Given $0 \le z_0 \in \mathcal{C}_c^\infty(I)$, $z \not\equiv 0$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ is a nonnegative function. Then, for any $\eta \in (0, \min\{\varepsilon, \|z_0\|_{L^\infty(I)}\})$, there exists a unique classical solution $z_{\varepsilon,\eta}$ of equation (9). Moreover, there is a positive constant $C(\beta,p)$ such that*

$$|\partial_x z_{\varepsilon,\eta}(x,\tau)| \le C.z_{\varepsilon,\eta}^{1-\frac{1}{\gamma}}(x,\tau)\left(\tau^{-\frac{1}{p}}\|z_0\|_\infty^{\frac{1+\beta}{p}} + M_f(z_0).\|z_0\|_\infty^{\frac{\beta}{p}} + M_{f'}(z_0).\|z_0\|_\infty^{\frac{1+\beta}{p}} + 1\right), \tag{10}$$

for $(x,\tau) \in I \times (0,\infty)$. Recall here $\quad M_g(u_0) = \left(\max_{0\le s\le 2\|u_0\|_\infty} |g(s)|\right)^{\frac{1}{p}}$.

Proof: The existence and uniqueness of solution, $z_{\varepsilon,\eta} \in \mathcal{C}^\infty(\overline{I} \times [0,\infty))$ is well-known (see, e.g. [16], [21], [29], [16] and [30]). For sake of brevity, let us drop dependence on ε, η in the notation of $z_{\varepsilon,\eta}$, and put

$$z = z_{\varepsilon,\eta}.$$

It is clear that η (resp. $\|z_0\|_{L^\infty(I)} + \eta$) is a sub-solution (resp. super-solution) of equation (9). Then, the comparison principle yields

$$\eta \le z \le \|z_0\|_{L^\infty(I)} + \eta \le 2\|z_0\|_{L^\infty(I)}, \quad \text{in } I \times (0,\infty). \tag{11}$$

For any $0 < \tau < T < \infty$, let us consider a test function $\xi(t) \in \mathcal{C}_c^\infty(0,\infty)$, $0 \le \xi(t) \le 1$ such that

$$\xi(t) = \begin{cases} 1, & \text{on } [\tau, T], \\ \\ 0, & \text{outside } (\frac{\tau}{2}, T + \frac{\tau}{2}). \end{cases}, \quad \text{and } |\xi_t| \le \frac{c_0}{\tau},$$

and put

$$z = \varphi(v) = v^\gamma, \quad w(x,t) = \xi(t)v_x^2.$$

Then, we have

$$w_t - aw_{xx} = \xi_t.v_x^2 + 2\xi v_x(v_t - av_{xx})_x - 2\xi a v_{xx}^2 + 2\xi a_x v_{xx}. \tag{12}$$

From the equation satisfied by z, we get

$$v_t - av_{xx} = a_x v_x + av_x^2 \frac{\varphi''}{\varphi'} - \frac{g_\varepsilon(\varphi)}{\varphi'} - \frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'},$$

Combining the last two equations provides us

$$w_t - aw_{xx} = \xi_t v_x^2 + 2\xi v_x \left(a_x v_x + av_x^2 \frac{\varphi''}{\varphi'} - \frac{g_\varepsilon(\varphi)}{\varphi'} - \frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'} \right)_x - 2\xi a v_{xx}^2 + 2\xi a_x v_{xx}.$$

Now, we define

$$L = \max_{\overline{I}\times[0,\infty)} \{w(x,t)\}.$$

If $L = 0$, then the conclusion (10) is trivial, and $|z_x(x,\tau)| = 0, \quad \text{in } I \times (0,\infty)$. If not we have $L > 0$, then the function w must attain its maximum at a point $(x_0,t_0) \in I \times (0,\infty)$ since $w(x,t) = 0$ on $\partial I \times (0,\infty)$ and $w(x,t)|_{t=0} = 0$. These facts lead to

$$\begin{cases} w_t(x_0,t_0) = w_x(x_0,t_0) = 0, \\ \text{and} \\ w_{xx}(x_0,t_0) \leq 0, \end{cases}$$

and $v_x(x_0,t_0) \neq 0$, so we get

$$w_x(x_0,t_0) = 0 \text{ if and only if } v_{xx}(x_0,t_0) = 0. \tag{13}$$

At the point (x_0,t_0), (12) and (13) provide us

$$0 \leq w_t - aw_{xx} = \xi_t v_x^2 + 2\xi v_x \left(a_{xx} v_x + a_x v_x^2 \frac{\varphi''}{\varphi'} + av_x^2 \left(\frac{\varphi''}{\varphi'}\right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x - \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x \right).$$

$$0 \leq \xi_t \xi^{-1} v_x^2 + 2v_x \left(a_{xx} v_x + a_x v_x^2 \frac{\varphi''}{\varphi'} + av_x^2 \left(\frac{\varphi''}{\varphi'}\right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x - \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x \right). \tag{14}$$

By using again (13), we obtain

$$a_x(z_x)(x_0,t_0) = (p-2) b^{\frac{p-4}{2}}(z_x)\varphi'\varphi'' v_x^3, \tag{15}$$

and

$$a_{xx}(z_x)(x_0,t_0) = (p-2)(p-4) b^{\frac{p-6}{2}}(z_x)(\varphi'.\varphi'')^2 v_x^6 + (p-2) b^{\frac{p-4}{2}}(z_x)(\varphi''^2 + \varphi'\varphi''') v_x^4. \tag{16}$$

Next, we have

$$\left(\frac{\varphi''}{\varphi'}\right)_x = \left(\frac{\varphi'''\varphi' - \varphi''^2}{\varphi'^2}\right) v_x = -(\gamma-1) v^{-2} v_x, \tag{17}$$

and

$$\begin{cases} v_x \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x = (g'_\varepsilon - g_\varepsilon \frac{\varphi''}{\varphi'^2}) v_x^2 = \left(\psi'_\varepsilon(\varphi) v^{-\beta} - (\beta + \frac{\gamma-1}{\gamma})\psi_\varepsilon(\varphi) v^{-(1+\beta)\gamma}\right) v_x^2, \\ \\ v_x \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x = \left((f\psi_\varepsilon)' - (f\psi_\varepsilon)\frac{\varphi''}{\varphi'^2}\right) v_x^2 = (f\psi_\varepsilon)' v_x^2 - f(\varphi(v)).\psi_\varepsilon(\varphi(v)).(\frac{\gamma-1}{\gamma}) v^{-\gamma} v_x^2. \end{cases}$$

Since $f, \psi_\varepsilon, \psi'_\varepsilon \geq 0$, and $0 \leq \psi_\varepsilon \leq 1$, we get

$$\begin{cases} v_x \left(\frac{g(\varphi)}{\varphi'}\right)_x \geq -(\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2, \\ v_x \left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x \geq f'(\varphi(v))\psi_\varepsilon v_x^2 - (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2. \end{cases} \tag{18}$$

Inserting (15), (16), (17), and (18), into (14) yields

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + \underbrace{(p-2)(p-4)b^{\frac{p-6}{2}}(\varphi'\varphi'')^2v_x^8 + (p-2)b^{\frac{p-4}{2}}(2\varphi''^2 + \varphi'\varphi''')v_x^6}_{\mathcal{B}} +$$

$$(\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2 - f'(\varphi(v))\psi_\varepsilon v_x^2 \geq (\gamma-1)v^{-2}a(z_x)v_x^4. \tag{19}$$

Next, we make a computation to handle $\mathcal{B}$

$$\begin{gathered} \mathcal{B} = (p-2)b^{\frac{p-6}{2}}(z_x)v_x^6\left((p-4)(\varphi'\varphi'')^2v_x^2 + (2\varphi''^2 + \varphi'\varphi''')b(z_x)\right) = \\ (p-2)\varphi'^2b^{\frac{p-6}{2}}(z_x)v_x^8\left((p-2)\varphi''^2 + \varphi'\varphi'''\right) + \eta^\alpha(p-2)(2\varphi''^2 + \varphi'\varphi''')b^{\frac{p-6}{2}}(z_x)v_x^6 = \\ \underbrace{(p-2)(p(\gamma-1)-\gamma)\gamma^2(\gamma-1)v^{2(\gamma-2)}\varphi'^2b^{\frac{p-6}{2}}(z_x)v_x^8}_{\mathcal{B}_1} + \underbrace{\eta^\alpha(p-2)\gamma^2(\gamma-1)(3\gamma-4)v^{2(\gamma-2)}b^{\frac{p-6}{2}}(z_x)v_x^6}_{\mathcal{B}_2} \end{gathered}$$

We observe that $\mathcal{B}_1 \leq 0$ since $p(\gamma-1)-\gamma < 0$, so we have

$$\mathcal{B} \leq \mathcal{B}_2. \tag{20}$$

By (19) and (20), we get

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2 - f'(\varphi(v))\psi_\varepsilon v_x^2 + \mathcal{B}_2 \geq (\gamma-1)v^{-2}a(z_x)v_x^4.$$

The fact that $b^{\frac{p-2}{2}}(.)$ is an increasing function since $p > 2$ leads to

$$a(z_x) = b^{\frac{p-2}{2}}(z_x) \geq (v_x^2\varphi'^2)^{\frac{p-2}{2}} = |v_x|^{p-2}\gamma^{p-2}v^{(\gamma-1)(p-2)}.$$

From the two last inequalities, we obtain

$$\begin{gathered} \frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2 - \\ f'(\varphi(v))\psi_\varepsilon v_x^2 + \mathcal{B}_2 \geq (\gamma-1)\gamma^{p-2}v^{(\gamma-1)(p-2)-2}|v_x|^{p+2}. \end{gathered}$$

By noting that $2-(\gamma-1)(p-2) = (1+\beta)\gamma$, we get

$$\begin{gathered} \frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2 - \\ f'(\varphi(v))\psi_\varepsilon v_x^2 + \mathcal{B}_2 \geq (\gamma-1)\gamma^{p-2}v^{-(1+\beta)\gamma}|v_x|^{p+2}. \end{gathered}$$

Multiplying both sides of the above inequality by $v^{(1+\beta)\gamma}$ yields

$$\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2+(\beta+\frac{\gamma-1}{\gamma})v_x^2+(\frac{\gamma-1}{\gamma})f(\varphi(v))v^{\beta\gamma}v_x^2-$$
$$f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma}v_x^2+v^{(1+\beta)\gamma}\mathcal{B}_2\geq(\gamma-1)\gamma^{p-2}|v_x|^{p+2}. \tag{21}$$

Now, we divide the study of inequality (21) in two cases:

(i) Case: $3\gamma-4\leq 0$.

We have $\mathcal{B}_2\leq 0$. It follows then from (21) that

$$(\gamma-1)\gamma^{p-2}|v_x|^{p+2}\leq\left(\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}+(\beta+\frac{\gamma-1}{\gamma})+(\frac{\gamma-1}{\gamma})f(\varphi(v))v^{\beta\gamma}-f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma}\right)v_x^2. \tag{22}$$

Remind that $z=\varphi(v)=v^\gamma$. Thus, we infer from (11) and (22)

$$|v_x(x_0,t_0)|^2\leq C_1\left(|\xi_t|\xi^{-1}(t_0)\|z_0\|_\infty^{1+\beta}+\|z_0\|_\infty^{\beta}.M_f^p(z_0)+\|z_0\|_\infty^{1+\beta}.M_{f'}^p(z_0)+1\right)^{\frac{2}{p}}, \tag{23}$$

where $C_1=C_1(\beta,p)>0$. Using Young's inequality in the right hand side of (23) deduces

$$|v_x(x_0,t_0)|^2\leq C_2\left(|\xi_t(t_0)|^{\frac{2}{p}}\xi^{-\frac{2}{p}}(t_0)\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}+\|z_0\|_\infty^{\frac{2\beta}{p}}.M_f^2(z_0)+\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}.M_{f'}^2(z_0)+1\right),$$

with $C_2=C_2(\beta,p)$, which implies

$$\begin{aligned}&w(x_0,t_0)=\xi(t_0)|v_x(x_0,t_0)|^2\leq\\&C_2.\xi(t_0)\left(|\xi_t(t_0)|^{\frac{2}{p}}\xi^{-\frac{2}{p}}(t_0)\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}+\|z_0\|_\infty^{\frac{2\beta}{p}}.M_f^2(z_0)+\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}.M_{f'}^2(z_0)+1\right)=\\&C_2\left(|\xi_t(t_0)|^{\frac{2}{p}}\xi^{1-\frac{2}{p}}(t_0)\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}+\xi(t_0).\|z_0\|_\infty^{\frac{2\beta}{p}}.M_f^2(z_0)+\xi(t_0).\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}.M_{f'}^2(z_0)+\xi(t_0)\right).\end{aligned}$$

Since $\xi(t)\leq 1$, $|\xi_t(t)|\leq\dfrac{c_0}{\tau}$ and $w(x_0,t_0)=\max\limits_{(x,t)\in I\times[0,\infty)}\{w(x,t)\}$, the last estimate induces

$$w(x,t)\leq C_2\left(\tau^{-\frac{2}{p}}\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}+\|z_0\|_\infty^{\frac{2\beta}{p}}.M_f^2(z_0)+\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}.M_{f'}^2(z_0)+1\right).$$

Thus, at time $t=\tau$ we have

$$w(x,\tau)=\xi(\tau).|v_x(x,\tau)|^2\overset{\xi(\tau)=1}{=}|v_x(x,\tau)|^2.$$

Then it follows from the last inequality

$$|v_x(x,\tau)|^2\leq C_2\left(\tau^{-\frac{2}{p}}\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}+\|z_0\|_\infty^{\frac{2\beta}{p}}.M_f^2(z_0)+\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}.M_{f'}^2(z_0)+1\right),$$

which implies

$$|z_x(x,\tau)|\leq C_3.z^{1-\frac{1}{\gamma}}\left(\tau^{-\frac{2}{p}}\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}+\|z_0\|_\infty^{\frac{2\beta}{p}}.M_f^2(z_0)+\|z_0\|_\infty^{\frac{2(1+\beta)}{p}}.M_{f'}^2(z_0)+1\right)^{\frac{1}{2}}.$$

Or

$$|z_x(x,\tau)| \le C_3.z^{1-\frac{1}{\gamma}} \left(\tau^{-\frac{1}{p}} \|z_0\|_{L^\infty(I)}^{\frac{(1+\beta)}{p}} + \|z_0\|_\infty^{\frac{\beta}{p}}.M_f(z_0) + \|z_0\|_\infty^{\frac{(1+\beta)}{p}}.M_{f'}(z_0) + 1 \right).$$

This inequality holds for any $\tau > 0$, so we get conclusion (10).

(ii) Case: $3\gamma - 4 > 0 \Longleftrightarrow p < 4(1-\beta)$.

Now $b^{\frac{p-6}{2}}(.)$ is a decreasing function, so we have

$$b^{\frac{p-6}{2}}(z_x) \le |z_x|^{\frac{p-6}{2}} = (v_x^2 \varphi'^2)^{\frac{p-6}{2}},$$

which implies

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \le \eta^\alpha (p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6} v^{2(\gamma-2)+(1+\beta)\gamma+(\gamma-1)(p-6)}|v_x|^p.$$

Note that $2(\gamma-2)+(1+\beta)\gamma+(\gamma-1)(p-6) = -2(\gamma-1)$. Then, we obtain

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \le \eta^\alpha (p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6} v^{-2(\gamma-1)}|v_x|^p. \tag{24}$$

A combination of (24) and (21) gives us

$$\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v_x^2 + (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{\beta\gamma}v_x^2 - f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma}v_x^2 +$$
$$\eta^\alpha(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{-2(\gamma-1)}|v_x|^p \ge (\gamma-1)\gamma^{p-2}|v_x|^{p+2}.$$

The fact $v = z^{\frac{1}{\gamma}} \ge \eta^{\frac{1}{\gamma}}$ implies $v^{-2(\gamma-1)} \le \eta^{-\frac{2(\gamma-1)}{\gamma}}$. Therefore, we get

$$|v_x(x_0,t_0)|^{p+2} \le C_4 \left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + 1 + f(\varphi(v))v^{\beta\gamma} - f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma} \right) v_x^2(x_0,t_0) +$$
$$C_4.\eta^{\alpha - \frac{2(\gamma-1)}{\gamma}}|v_x(x_0,t_0)|^p,$$

with $C_4 = C_4(\beta,p) > 0$.

Now, if $|v_x(x_0,t_0)| < 1$, then we have $\xi(t_0)|v_x(x_0,t_0)|^2 < 1$, likewise $w(x,t) \le 1$, in $I \times (0,\infty)$. Thus, the conclusion (10) follows immediately. If not, we have $|v_x(x_0,t_0)|^p \le |v_x(x_0,t_0)|^{p+2}$, thereby proves

$$|v_x(x_0,t_0)|^{p+2} \le C_4 \left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + f(\varphi(v))v^{\beta\gamma} - f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma} + 1 \right) v_x^2(x_0,t_0) +$$
$$C_4.\eta^{\alpha - \frac{2(\gamma-1)}{\gamma}}|v_x(x_0,t_0)|^{p+2},$$

or

$$\left(1 - C_4.\eta^{\alpha-\frac{2(\gamma-1)}{\gamma}}\right)|v_x(x_0,t_0)|^{p+2} \le C_4 \left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + f(\varphi(v))v^{\beta\gamma} - f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma} + 1 \right) v_x^2(x_0,t_0).$$

Since $\alpha > \frac{2(\gamma-1)}{\gamma}$ and $\eta \to 0^+$, there exists a positive constant $C_5 = C_5(\beta,p) > 0$ such that

$$|v_x(x_0,t_0)|^{p+2} \le C_5 \left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + f(\varphi(v))v^{\beta\gamma} - f'(\varphi(v))\psi_\varepsilon v^{(1+\beta)\gamma} + 1 \right) v_x^2(x_0,t_0).$$

This inequality is just a version of (22). By the same analysis as in **(i)**, we also obtain estimate (10). This puts an end to the proof of Lemma 13. $\square$

Remark 14 *If f is only a global Lipschitz function with its Lipschitz constant C_f, then by Rademacher's theorem (see also in [20]), estimate* (10) *becomes*

$$|\partial_x z_{\varepsilon,\eta}(x,\tau)| \leq C.z_{\varepsilon,\eta}^{1-\frac{1}{\gamma}}(x,\tau)\left(\tau^{-\frac{1}{p}}\|z_0\|_\infty^{\frac{1+\beta}{p}} + M_f(z_0).\|z_0\|_\infty^{\frac{\beta}{p}} + C_f^{\frac{1}{p}}.\|z_0\|_\infty^{\frac{1+\beta}{p}} + 1\right), \qquad (25)$$

for $(x,\tau) \in I \times (0,\infty)$.

If f in Lemma 13 is a nondecreasing function, then we can relax the term containing $M_{f'}(.)$ in estimate (10).

Lemma 15 *Assume that $f \in \mathcal{C}^1(\mathbb{R})$ is a nondecreasing function. Then, estimate* (10) *can be relaxed as follows*

$$|\partial_x z_{\varepsilon,\eta}(x,\tau)| \leq C.z_{\varepsilon,\eta}^{1-\frac{1}{\gamma}}(x,\tau)\left(\tau^{-\frac{1}{p}}\|z_0\|_\infty^{\frac{1+\beta}{p}} + M_f(z_0).\|z_0\|_\infty^{\frac{\beta}{p}} + 1\right), \qquad (26)$$

for $(x,\tau) \in I \times (0,\infty)$. Note that $M_f^p(z_0) = f(2\|z_0\|_\infty)$ since f is nondecreasing.

Proof: The proof of this Lemma is most likely to the one of Lemma 13. In fact, we just make a slight change in (18) in order to remove the term involving f'. Recall here (18):

$$v_x\left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x \geq f'(\varphi(v))\psi_\varepsilon v_x^2 - (\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2.$$

Since $f', \psi_\varepsilon \geq 0$, we obtain

$$v_x\left(\frac{f(\varphi)\psi_\varepsilon(\varphi)}{\varphi'}\right)_x \geq -(\frac{\gamma-1}{\gamma})f(\varphi(v))v^{-\gamma}v_x^2.$$

After that, we just repeat the proof of Lemma 13 without the term containing f'. Thus, we get estimate (26). $\square$

Remark 16 *We can also relax the assumption $f \in \mathcal{C}^1(\mathbb{R})$ in Lemma 15 by considering the standard regularization of f, i.e, $f_n = f * \varrho_n \in \mathcal{C}^1(\mathbb{R})$, where $\{\varrho_n\}_{n\geq 1}$ is the sequence of mollifier functions.*

Next, we shall show that $z_{\varepsilon,\eta}$ is a Lipschitz function on $I \times (\tau,\infty)$ with a Lipschitz constant C being independent of ε, η.

Proposition 17 *Assume f as in Lemma 13. Let $z_{\varepsilon,\eta}$ be the solution of equation* (9) *above. Then, for any $\tau > 0$ there is a positive constant $C(\beta, p, \tau, |I|, \|z_0\|_\infty)$ such that*

$$|z_{\varepsilon,\eta}(x,t) - z_{\varepsilon,\eta}(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{2}}\right), \quad \forall x,y \in \overline{I}, \quad \forall t,s \geq \tau. \qquad (27)$$

Proof: We first extend $z_{\varepsilon,\eta}$ by η outside I, (still denoted as $z_{\varepsilon,\eta}$). To simplify the notation, we denote again $z = z_{\varepsilon,\eta}$.

Fix $\tau > 0$. Multiplying equation (9) by $\partial_t z$, and using integration by parts yield

$$\int_s^t \int_I |\partial_t z|^2 + a(z_x) z_x \partial_t z_x + g_\varepsilon(z)\partial_t z + f(z)\psi_\varepsilon(z)\partial_t z \; dx d\sigma = 0, \quad \text{for } t > s \geq \tau. \tag{28}$$

We observe that

$$a(z_x) z_x \partial_t z_x = \left(|z_x|^2 + \eta^\alpha\right)^{\frac{p-2}{2}} . \frac{1}{2}\partial_t(|z_x|^2) = \frac{1}{p}\partial_t(|z_x|^2 + \eta^\alpha)^{\frac{p}{2}}.$$

By this fact, we deduce from equation (28)

$$\int_s^t \int_I |\partial_t z(x,\sigma)|^2 dx d\sigma \leq \int_I \frac{1}{p}\left(|z_x(x,s)|^2 + \eta^\alpha\right)^{\frac{p}{2}} dx + \int_I G_\varepsilon(z(x,s))dx + \int_I H_\varepsilon(z(x,s))dx,$$

with

$$\begin{cases} G_\varepsilon(r) = \displaystyle\int_0^r g_\varepsilon(s)ds \leq \int_0^r s^{-\beta} ds = \frac{r^{1-\beta}}{1-\beta}, \\ \\ H_\varepsilon(r) = \displaystyle\int_0^r f(s)\psi_\varepsilon(s)ds \leq r f(r), \quad \text{since } f \text{ is nondecreasing, and } \psi_\varepsilon \leq 1. \end{cases}$$

Then, we obtain

$$\int_s^t \int_I |\partial_t z(x,\sigma)|^2 dx d\sigma \leq \frac{1}{p}\int_I \left(|z_x(x,s)|^2 + \eta^\alpha\right)^{\frac{p}{2}} dx + \frac{1}{1-\beta}\int_I z(x,s)^{1-\beta} dx + \int_I z(x,s) f(z(x,s))dx,$$

or

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \overset{(11)}{\leq} \frac{1}{p}\int_I \left(\|z_x(s)\|_\infty^2 + \eta^\alpha\right)^{\frac{p}{2}} dx + \frac{1}{1-\beta}\int_I \left(2\|z_0\|_\infty\right)^{1-\beta} dx + \int_I 2\|z_0\|_\infty . M_f^p(z_0) dx.$$

We apply Young's inequality to the first term in the right hand side to get

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq C_6 \left(\|z_x(s)\|_\infty^p + \|z(s)\|_\infty^{1-\beta} + \|z_0\|_\infty . M_f^p(z_0)\right) + O(\eta), \tag{29}$$

with $C_6 = C_6(\beta, p, |I|)$, and $\lim\limits_{\eta \to 0} O(\eta) = 0$.
By (10) (or (26)), and (29), there is a constant $C_7(\beta, p, \tau, |I|, \|z_0\|_\infty) > 0$ such that

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq C_7, \quad \forall t > s \geq \tau. \tag{30}$$

Estimate (30) means that $\|\partial_t z_{\varepsilon,\eta}\|_{L^2(I\times(s,t))}$ is bounded by a constant, which is independent of ε and η.

Next, for any $x, y \in I$ and for $t > s \geq \tau$, we set

$$r = |x - y| + |t - s|^{\frac{1}{2}}.$$

According to the Mean Value Theorem, there is a real number $\bar{x} \in I_r(y)$ such that

$$|\partial_t z(\bar{x}, \sigma)|^2 = \frac{1}{|I_r(y)|} \int_{I_r(y)} |\partial_t z(l, \sigma)|^2 dl = \frac{1}{2r} \int_{I_r(y) \cap I} |\partial_t z(l, \sigma)|^2 dl \leq \frac{1}{2r} \int_I |\partial_t z(l, \sigma)|^2 dl \quad (31)$$

(Note that $\partial_t z(., t) = 0$ outside I).
Next, we have from Holder's inequality

$$|z(\bar{x}, t) - z(\bar{x}, s)|^2 \leq (t - s) \int_s^t |\partial_t z(\bar{x}, \sigma)|^2 d\sigma \overset{(31)}{\leq} \frac{(t - s)}{2r} \int_s^t \int_I |\partial_t z(l, \sigma)|^2 dl d\sigma,$$

or

$$|z(\bar{x}, t) - z(\bar{x}, s)|^2 \leq \frac{1}{2}(t - s)^{\frac{1}{2}} \int_s^t \int_I |\partial_t z(l, \sigma)|^2 dl d\sigma. \quad (32)$$

From (30) and (32), we obtain

$$|z(\bar{x}, t) - z(\bar{x}, s)|^2 \leq \frac{1}{2} C_7 (t - s)^{\frac{1}{2}}, \quad \forall t > s \geq \tau. \quad (33)$$

Now, it is sufficient to show (27). Indeed, we have the triangular inequality

$$\begin{aligned} |z(x, t) - z(y, s)| \leq |z(x, t) - z(y, t)| + |z(y, t) - z(y, s)| \leq |z(x, t) - z(y, t)| + \\ |z(y, t) - z(\bar{x}, t)| + |z(\bar{x}, t) - z(\bar{x}, s)| + + |z(\bar{x}, s) - z(y, s)|, \end{aligned}$$

where $\bar{x} \in I_r(y)$ is above. Then, the conclusion (27) just follows from (33), gradient estimates (10), (26) and the Mean Value Theorem. Or, we get the proof of the above Proposition. □

Remark 18 *The result of the above Proposition still holds for the case where f is as in Lemma 15 or Remark 14.*

Note that the estimates in the proof of Lemma 13 (resp. Lemma 15) and Proposition 17 are independent of η, ε. This observation allows us to pass to the limit as $\eta \to 0$ in order to get gradient estimates (10) (resp. (25), (26)) for the following problem

$$(P_\varepsilon) \begin{cases} \partial_t z - \partial_x \left(|\partial_x z|^{p-2} \partial_x z \right) + g_\varepsilon(z) + f(z) \psi_\varepsilon(z) = 0 & \text{in } I \times (0, \infty), \\ z(-l, t) = z(l, t) = 0 & t \in (0, \infty), \\ z(x, 0) = z_0(x) & \text{on } I. \end{cases} \quad (34)$$

Theorem 19 *Let $0 \leq z_0 \in \mathcal{C}_c^\infty(I)$, $z_0 \neq 0$. Assume f as in Lemma 13. Then, there exists a unique bounded weak solution z_ε of problem (P_ε). Furthermore, z_ε also fulfills estimate (10), and the regularity result (27).*

Remark 20 *The result of Theorem 19 also holds if f is assumed as in Lemma 15 (resp. Remark 14). Moreover, z_ε fulfills estimate (26) (resp. (25)).*

Proof: The existence and uniqueness of solution of problem (P_ε) is a classical result (see e.g [29], [16], and [30]). Thanks to Lemma 13 and the uniqueness result, Theorem 19 follows by passing $\eta \to 0$. $\square$

3 Existence of a maximal solution

In this section, we focus on the proof of Theorem 3 (Theorem 2 is proved similarly). Then, we divide the proof of Theorem 3 into three steps. In the first step, we prove the existence and uniqueness of solution u_ε of problem (P_ε) with initial data $u_0 \in L^1(\Omega)$. Moreover, we prove an estimate for $|\partial_x u_\varepsilon|$ involving $u_\varepsilon^{1-1/\gamma}$ and $\|u_0\|_{L^1(I)}$, see Theorem 21 below. After that, we will go to the limit as $\varepsilon \to 0$ in order to get $u_\varepsilon \to u$, a solution of equation (1). Finally, the conclusion that u is a maximal solution is proved in Proposition 24 below.

We first have the following result:

Theorem 21 *Let $0 \le u_0 \in L^1(I)$, $u_0 \neq 0$. Assume that f satisfies (H_2). Then, there exists a unique weak solution u_ε of problem (P_ε) with initial data u_0. Moreover, for any $\tau > 0$, there is a constant $C = C(\beta, p, |I|) > 0$ such that*

$$|\partial_x u_\varepsilon(x,t)| \le C.u_\varepsilon^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + \tau^{-\frac{\beta}{p}}\|u_0\|_{L^1(I)}^{\frac{\beta}{\lambda}}.m_f(\tau,u_0) + 1\right), \tag{35}$$

for a.e $(x,t) \in (\tau,\infty)$, recall here $m_f(t,u_0) = f^{\frac{1}{p}}\left(2C(p,|I|).t^{-\frac{1}{\lambda}}\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}\right)$.

As a consequence of (35) and Proposition 17, u_ε is a Lipschitz function on $\overline{I} \times [t_1, t_2]$, for any $0 < t_1 < t_2 < \infty$. Moreover, the Lipschitz constant of u_ε is independent of ε.

Proof: **(i) Uniqueness:** The uniqueness result follows from the Lemma below.

Lemma 22 *Let v_1 (resp. v_2) be a weak sub-solution (resp. super solution) of equation (34). Then, we have*

$$v_1 \le v_2, \quad \text{in } I \times (0,\infty).$$

Proof: We skip the proof of Lemma 22 and give its proof in the Appendix. $\square$

(ii) Existence: We regularize initial data u_0 by considering a sequence, $\{u_{0,n}\}_{n\ge1} \subset \mathcal{C}_c^\infty(I)$ such that $u_{0,n} \overset{n\to\infty}{\longrightarrow} u_0$ in $L^1(I)$, and $\|u_{0,n}\|_{L^1(I)} \le \|u_0\|_{L^1(I)}$. Let $u_{\varepsilon,n}$ be a unique (weak) solution of equation (34) with initial data $u_{0,n}$ (see e.g [16], [30], and [29]). We will show that $u_{\varepsilon,n}$ converges to u_ε, which is a solution of equation (34) with initial data u_0.

First of all, we observe that $u_{\varepsilon,n}$ is a sub-solution of the following equation

$$\begin{cases} \partial_t v_n - \left(|\partial_x v_n|^{p-2}\partial_x v_n\right)_x = 0 & \text{in } I \times (0,\infty), \\ v_n(-l,t) = v_n(l,t) = 0 & \forall t \in (0,\infty), \\ v_n(x,0) = u_{0,n}(x) & \text{in } I, \end{cases} \tag{36}$$

thereby

$$u_{\varepsilon,n} \leq v_n, \quad \text{in } I \times (0,\infty). \tag{37}$$

Moreover, there is a positive constant $C(p,|I|)$ such that

$$\|v_n(.,t)\|_\infty \leq C(p,|I|).t^{-\frac{1}{\lambda}}\|v_n(0)\|_{L^1(I)}^{\frac{p}{\lambda}} \leq C(p,|I|).t^{-\frac{1}{\lambda}}\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}, \quad \forall t>0, \tag{38}$$

(see, e.g Theorem 4.3, [12]), so we get from (37) and (38)

$$\|u_{\varepsilon,n}(.,t)\|_\infty \leq C(p,|I|).t^{-\frac{1}{\lambda}}\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}, \quad \forall t>0. \tag{39}$$

For any $\tau > 0$, inequality (39) means that $\|u(t)\|_\infty$ is bounded for $t \geq \tau$. Then, we can apply Theorem 19 to $u_{\varepsilon,n}$ by considering $u_{\varepsilon,n}(\tau)$ as the initial data in order to get

$$|\partial_x u_{\varepsilon,n}(x,t)| \leq C(\beta,p)u_{\varepsilon,n}^{1-\frac{1}{\gamma}}(x,t)\left((t-\tau)^{-\frac{1}{p}}\|u_{\varepsilon,n}(\tau)\|_\infty^{\frac{1+\beta}{p}} + \|u_{\varepsilon,n}(\tau)\|_\infty^{\frac{\beta}{p}}.f^{\frac{1}{p}}(2\|u_{\varepsilon,n}(\tau)\|_\infty)+1\right),$$

for a.e $(x,t) \in I \times (\tau,\infty)$. In particular, we obtain for a.e $(x,t) \in I \times (2\tau,\infty)$

$$|\partial_x u_{\varepsilon,n}(x,t)| \leq Cu_{\varepsilon,n}^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{1}{p}}\|u_{\varepsilon,n}(\tau)\|_\infty^{\frac{1+\beta}{p}} + \|u_{\varepsilon,n}(\tau)\|_\infty^{\frac{\beta}{p}}.f^{\frac{1}{p}}\left(2\|u_{\varepsilon,n}(\tau)\|_\infty\right)+1\right). \tag{40}$$

Recall $m_f(t,u_0) = f^{\frac{1}{p}}\left(2C(p,|I|).t^{-\frac{1}{\lambda}}\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}\right)$. Combining (39) and (40) yields

$$|\partial_x u_{\varepsilon,n}(x,t)| \leq C.u_{\varepsilon,n}^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + \tau^{-\frac{\beta}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{\beta}{\lambda}}.m_f(\tau,u_0)+1\right), \tag{41}$$

for a.e $(x,t) \in (2\tau,\infty)$. In view of (41), $|\partial_x u_{\varepsilon,n}(x,t)|$ is bounded on $I \times [2\tau,\infty)$ by a positive constant being independent of ε and n. Thanks to Proposition 17, we have

$$|u_{\varepsilon,n}(x,t) - u_{\varepsilon,n}(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{2}}\right), \quad \forall x,y \in \overline{I},\ \forall t,s > 2\tau. \tag{42}$$

Note that C in (42) only depends on $\beta, p, \tau, |I|$, and $\|u_0\|_{L^1(I)}$ (instead of $\|u_0\|_{L^\infty(I)}$ as in Proposition 17).

Now, we can pass to the limit as $n \to \infty$. To avoid relabeling after any passage to the limit, we want to keep the same label. Then, we observe that (42) allows us to apply the Ascoli-Arzela Theorem to $u_{\varepsilon,n}$, so there is a subsequence of $\{u_{\varepsilon,n}\}_{n\geq 1}$ such that for any $2\tau < t_1 < t_2 < \infty$

$$u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} u_\varepsilon, \quad \text{uniformly on compact set } \overline{I} \times [t_1,t_2]. \tag{43}$$

It follows from the diagonal argument that there is a subsequence of $\{u_{\varepsilon,n}\}_{n\geq 1}$ such that

$$u_{\varepsilon,n}(x,t) \overset{n\to\infty}{\longrightarrow} u_\varepsilon(x,t), \quad \text{pointwise in } \overline{I} \times (0,\infty).$$

Thus, it is clear that u_ε also fulfills the a priori bound (39) and the Lipschitz continuity (42).

After that, we show that for any $T \in (0,\infty)$

$$g_\varepsilon(u_{\varepsilon,n}) \overset{n\to\infty}{\longrightarrow} g_\varepsilon(u_\varepsilon), \quad \text{in } L^1(I \times (0,T)). \tag{44}$$

In fact, $g_\varepsilon(.)$ is a global Lipschitz function, and it is bounded by $\varepsilon^{-\beta}$. Therefore, the Dominated Convergence Theorem yields the conclusion (44).

Next, we claim that for any $0 < t_1 < t_2 < \infty$

$$f(u_{\varepsilon,n})\psi_\varepsilon(u_{\varepsilon,n}) \overset{n\to\infty}{\longrightarrow} f(u_\varepsilon)\psi_\varepsilon(u_\varepsilon), \quad \text{in } L^1(I \times (t_1, t_2)). \tag{45}$$

According to (39) and the fact $f \in \mathcal{C}(\mathbb{R})$, we observe that $f(u_{\varepsilon,n}(x,t))$ is bounded on $I \times (t_1, \infty)$ by a constant not depending on ε, n. By applying Dominated Convergence Theorem, we get claim (45).

Besides, the contraction of L^1-norm gives us

$$\|g_\varepsilon(u_{\varepsilon,n})\|_{L^1(I\times(0,\infty))}, \quad \|f(u_{\varepsilon,n})\psi_\varepsilon(u_{\varepsilon,n})\|_{L^1(I\times(0,\infty))} \le \|u_0\|_{L^1(I)}. \tag{46}$$

It follows from (46), (45), and (44) that

$$\|g_\varepsilon(u_\varepsilon)\|_{L^1(I\times(0,\infty))}, \quad \|f(u_\varepsilon)\psi_\varepsilon(u_\varepsilon)\|_{L^1(I\times(0,\infty))} \le \|u_0\|_{L^1(I)}. \tag{47}$$

Next, we show that there is a subsequence of $\{u_{\varepsilon,n}\}_{n\ge1}$ such that

$$\partial_x u_{\varepsilon,n}(x,t) \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty). \tag{48}$$

In order to prove this, we borrow a result of L. Boccardo and F. Murat, [5] and [4], the so called *almost everywhere convergence of the gradients*. In fact, thanks to (46), (41), and (43), we can imitate the proof in [4], or [5] to get

$$\partial_x u_{\varepsilon,n}(x,t) \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in I \times (t_1,t_2),$$

up to a subsequence, for any $0 < t_1 < t_2$. Then, the claim (48) just follows from the diagonal argument. As a consequence, u_ε also fulfills estimate (41), and we have for any $0 < t_1 < t_2$

$$\partial_x u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon, \quad \text{in } L^q(I \times (t_1,t_2)), \quad \text{for any } q \ge 1. \tag{49}$$

By (49), (44), and (45), we observe that u_ε satisfies equation (1) in the weak sense. Then, it remains to show that

$$u_\varepsilon \in \mathcal{C}([0,T]; L^1(I)), \quad \text{for any } T \in (0,\infty). \tag{50}$$

Let us set

$$\begin{aligned}
T_k(u) &= \begin{cases} u, & \text{if } |u| \le k, \\ k.sign(u), & \text{if } |u| > k, \end{cases} \text{ and} \\
S_k(u) &= \int_0^u T_k(s)ds = \frac{1}{2}|u|^2\chi_{\{|u|\le k\}} + k(|u| - \frac{1}{2}k)\chi_{\{|u|>k\}}.
\end{aligned}$$

We consider the difference between two equations satisfied by $u_{\varepsilon,n}$ and $u_{\varepsilon,m}$:

$$\begin{aligned}
&\partial_t(u_{\varepsilon,n} - u_{\varepsilon,m}) - \partial_x\left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n}\right) + \partial_x\left(|\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m}\right) \\
&g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m}) + f(u_{\varepsilon,n})\psi_\varepsilon(u_{\varepsilon,n}) - f(u_{\varepsilon,m})\psi_\varepsilon(u_{\varepsilon,m}) = 0.
\end{aligned}$$

Multiplying the above equation with $T_1(u_{\varepsilon,n}-u_{\varepsilon,m})$, and integrating on $I\times(0,t)$ yields

$$\int_I S_1\left(u_{\varepsilon,n}-u_{\varepsilon,m}\right)(t)dx+\int_0^t\int_I\left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n}-|\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m}\right)(\partial_x u_{\varepsilon,n}-\partial_x u_{\varepsilon,m})dxds$$
$$+\int_0^t\int_I\left(g_\varepsilon(u_{\varepsilon,n})-g_\varepsilon(u_{\varepsilon,m})\right)T_1(u_{\varepsilon,n}-u_{\varepsilon,m})dxds+$$
$$\int_0^t\int_I\left(f(u_{\varepsilon,n})\psi_\varepsilon(u_{\varepsilon,n})-f(u_{\varepsilon,m})\psi_\varepsilon(u_{\varepsilon,m})\right)T_1(u_{\varepsilon,n}-u_{\varepsilon,m})dxds=\int_I S_1\left(u_{\varepsilon,n}-u_{\varepsilon,m}\right)(0)dx.$$

By the monotone of p-Laplacian operator and the monotone of the function $f\psi_\varepsilon$, we have

$$\int_I S_1\left(u_{\varepsilon,n}-u_{\varepsilon,m}\right)(t)dx\leq\int_I|u_{0,n}-u_{0,m}|dx+\int_0^t\int_I|g_\varepsilon(u_{\varepsilon,n})-g_\varepsilon(u_{\varepsilon,m})|dxds\overset{(44)}{=}o(n,m),$$

where $\lim\limits_{n,m\to\infty}o(n,m)=0$. Moreover, we have from the formula of $S_1(.)$ and Holder's inequality

$$\int_I|u_{\varepsilon,n}-u_{\varepsilon,m}|(x,t)dx\leq\sqrt{2|I|\int_I S_1(u_{\varepsilon,n}-u_{\varepsilon,m})(x,t)dx}+2\int_I S_1(u_{\varepsilon,n}-u_{\varepsilon,m})(x,t))dx,\quad\forall t>0.$$

Combining the two last inequalities yields

$$\int_I|u_{\varepsilon,n}-u_{\varepsilon,m}|(x,t)dx\leq C(|I|).\left(\sqrt{o(n,m)}+o(n,m)\right),\quad\forall t>0.\tag{51}$$

Thus, $\{u_{\varepsilon,n}\}_{n\geq1}$ is a Cauchy sequence in $\mathcal{C}([0,T];L^1(I))$, or we get (50). This puts an end to the proof of Theorem 21. □

In the second step, we will pass to the limit as $\varepsilon\to0$. Let us first claim that $\{u_\varepsilon\}_{\varepsilon>0}$ is a non-decreasing sequence, so there is a nonnegative function u such that $u_\varepsilon(x,t)\downarrow u(x,t)$ as $\varepsilon\to0$. Indeed, for any $\varepsilon>\varepsilon'>0$, it is clear that $g_{\varepsilon'}(s)\geq g_\varepsilon(s)$, and $\psi_{\varepsilon'}(s)\geq\psi_\varepsilon(s)$ for $s\in\mathbb{R}$. Therefore, u_ε is a super-solution of equation satisfied by $u_{\varepsilon'}$, so Lemma 22 yields

$$u_\varepsilon(x,t)\geq u_{\varepsilon'}(x,t),\quad\text{in } I\times(0,\infty),$$

or the claim follows. We would like to emphasize that the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ will be intensively used in what follows, although one can utilize Ascoli-Azela Theorem to show that $u_\varepsilon\to u$. Note that u is also a Lipschitz function on $\overline{I}\times[t_1,t_2]$, for any $0<t_1<t_2$.

To be similar to (48), we obtain $\partial_x u_\varepsilon\overset{\varepsilon\to0}{\longrightarrow}\partial_x u$, for a.e $(x,t)\in I\times(0,\infty)$. As a result, u_x fulfills estimate (41), and

$$\partial_x u_\varepsilon\overset{\varepsilon\to0}{\longrightarrow}\partial_x u,\quad\text{in } L^q(I\times(t_1,t_2)),\quad\forall q\geq1,\tag{52}$$

Next, let us show that

$$u^{-\beta}\chi_{\{u>0\}}\in L^1(I\times(0,\infty)).\tag{53}$$

From (45), applying Fatou's Lemma deduces that there is a function $\Phi\in L^1(I\times(0,\infty))$ such that

$$\liminf_{\varepsilon\to0}g_\varepsilon(u_\varepsilon)=\Phi,\quad\text{in } L^1(I\times(0,\infty)).\tag{54}$$

The monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ ensures $\quad g_\varepsilon(u_\varepsilon)(x,t) \geq g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t)$, so

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon)(x,t) = \Phi \geq u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for a.e } (x,t) \in I\times(0,\infty). \tag{55}$$

Thus, conclusion (53) just follows from (54) and (55). Actually, we will show at the end of this step that

$$\liminf_{\varepsilon\to 0} g_\varepsilon(u_\varepsilon) = \Phi = u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(I\times(0,\infty)). \tag{56}$$

Let us emphasize that (56) implies the conclusion

$$u \in \mathcal{C}([0,\infty); L^1(I)), \tag{57}$$

by following the proof of (50).

At the moment, we demonstrate that u must satisfy equation (1) in the sense of distribution. For any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, $\phi \in \mathcal{C}_c^\infty(I\times(0,\infty))$, in the equation satisfied by u_ε. Then, using integration by parts yields

$$\int_{Supp(\phi)} -\Psi_\eta(u_\varepsilon)\phi_t + \frac{1}{\eta}|\partial_x u_\varepsilon|^p\psi'(\frac{u_\varepsilon}{\eta})\phi + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon.\phi_x.\psi_\eta(u_\varepsilon)+$$
$$g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi + f(u_\varepsilon)\psi_\eta(u_\varepsilon).\psi_\varepsilon(u_\varepsilon)\phi \; dxds = 0,$$

with $\quad \Psi_\eta(u) = \displaystyle\int_0^u \psi_\eta(s)ds$. There is no problem of going to the limit as $\varepsilon \to 0$ in the indicated equation, so we have

$$\int_{Supp(\phi)} -\Psi_\eta(u)\phi_t + \frac{1}{\eta}|u_x|^p\psi'(\frac{u}{\eta})\phi + |u_x|^{p-2}u_x.\phi_x.\psi_\eta(u) + u^{-\beta}\psi_\eta(u)\phi + f(u)\psi_\eta(u)\phi \; dxds = 0.$$

Next, we go to the limit as $\eta \to 0$ in the above equation. From (39), (41), (52), and (53), it is not difficult to verify

$$\begin{cases} \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} \Psi_\eta(u)\phi_t \; dxds = \int_{Supp(\phi)} u.\phi_t \; dxds, \\[2ex] \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} |u_x|^{p-2}u_x.\phi_x.\psi_\eta(u) \; dxds = \int_{Supp(\phi)} |u_x|^{p-2}u_x.\phi_x \; dxds, \\[2ex] \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} u^{-\beta}\psi_\eta(u)\phi \; dxds = \int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi \; dxds, \\[2ex] \lim\limits_{\eta\to 0}\displaystyle\int_{Supp(\phi)} f(u)\psi_\eta(u)\phi \; dxds = \int_{Supp(\phi)} f(u)\phi \; dxds. \end{cases} \tag{58}$$

Note that the assumption $f(0) = 0$ is used in the final equality of (58). While, we have

$$\lim_{\eta\to 0}\int_{Supp(\phi)} \frac{1}{\eta}|\partial_x u|^p\psi'(\frac{u}{\eta})\phi dxds = 0. \tag{59}$$

In fact, since u satisfies estimate (41), we have

$$\frac{1}{\eta}\int_{Supp(\phi)}|\partial_x u|^p|\psi'(\frac{u}{\eta}).\phi|dxds \leq C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{1-\beta}dxds$$
$$\leq 2C\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}dxds,$$

where the constant $C>0$ merely depends on β, p, $Supp(\phi)$, $\|u_0\|_{L^1(I)}$. Moreover, $u^{-\beta}\chi_{\{u>0\}}$ is integrable on $I\times(0,\infty)$ by (53). Then, we obtain

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}}u^{-\beta}dxds = 0,$$

which implies the conclusion (59). A combination of (58) and (59) deduces

$$\int_{Supp(\phi)}\left(-u\phi_t + |u_x|^{p-2}u_x\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi\right)dxds = 0. \quad (60)$$

In other words, u satisfies equation (1) in $\mathcal{D}'(I\times(0,\infty))$.

As mentioned above, we prove (56) now. The fact that u_ε is a weak solution of (34) gives us

$$\int_{Supp(\phi)}\left(-u_\varepsilon\phi_t + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon\partial_x\phi + g_\varepsilon(u_\varepsilon)\phi + f(u_\varepsilon)\psi_\varepsilon(u_\varepsilon)\phi\right)dxds = 0,$$

for $\phi\in\mathcal{C}_c^\infty(I\times(0,\infty))$, $\phi\geq 0$. Letting $\varepsilon\to 0$ induces

$$\int_{Supp(\phi)}\left(-u\phi_t + |u_x|^{p-2}u_x\phi_x\right)dxds + \lim_{\varepsilon\to 0}\int_{Supp(\phi)}g_\varepsilon(u_\varepsilon)\phi\ dxds + \int_{Supp(\phi)}f(u)\phi\ dxds = 0. \quad (61)$$

By (60) and (61), we get

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_I g_\varepsilon(u_\varepsilon)\phi\ dxds = \int_0^\infty\int_I u^{-\beta}\chi_{\{u>0\}}\phi\ dxds. \quad (62)$$

According to (54), (62) and Fatou's Lemma, we obtain

$$\int_0^\infty\int_I u^{-\beta}\chi_{\{u>0\}}\phi dxds \geq \int_0^\infty\int_I \Phi\phi dxds, \quad \forall\phi\in\mathcal{C}_c^\infty(I\times(0,\infty)), \phi\geq 0.$$

The last inequality and (55) yield

$$u^{-\beta}\chi_{\{u>0\}} = \Phi, \quad \text{a.e in } I\times(0,\infty).$$

Thereby, we get (56). In conclusion, u is a weak solution of equation (1).

Remark 23 *The reader should note that* (60) *is not sufficient to conclude that u is a weak solution of equation* (1) *according to Definition* 1. *Thus, it is necessary to prove* (56)*, thereby proves* (57).

We end this Section by proving that u is the maximal solution of equation (1).

Proposition 24 *Let v be any weak solution of equation* (1)*. Then, we have*

$$v(x,t)\leq u(x,t), \quad \textit{for a.e } (x,t)\in I\times(0,\infty).$$

Proof: For any $\varepsilon > 0$, we observe that

$$g_\varepsilon(v) \le v^{-\beta}\chi_{\{v>0\}}, \text{and } f(v).\psi_\varepsilon(v) \le f(v).$$

Thus, we get

$$\partial_t v - \left(|v_x|^{p-2}v_x\right)_x + g_\varepsilon(v) + f(v).\psi_\varepsilon(v) \le \partial_t v - \left(|v_x|^{p-2}v_x\right)_x + v^{-\beta}\chi_{\{v>0\}} + f(v) = 0,$$

which implies that v is a sub-solution of equation (P_ε). Thanks to Lemma 22, we get

$$v(x,t) \le u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty).$$

Letting $\varepsilon \to 0$ yields the result. This puts an end to the proof of Theorem 3. □

If f is a global Lipschitz function, then we have

Theorem 25 *Let $0 \le u_0 \in L^1(I)$, $u_0 \ne 0$. Assume that f is a global Lipschitz function with Lipschitz constant C_f, and $f(0) = 0$. Then there exists a maximal weak solution u of equation (1). Furthermore, we have*

For any $\tau > 0$, there exist two positive constants $C_1(\beta, p, |I|)$ and $C_2(p, |I|)$ such that

$$|u_x(x,t)| \le C_1.u^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + \tau^{-\frac{\beta}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{\beta}{\lambda}}.M_f(u(\tau)) + C_f^{\frac{1}{p}}.\tau^{-\frac{1+\beta}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + 1\right), \tag{63}$$

for a.e $(x,t) \in I \times (\tau,\infty)$. Note that $M_f(u(\tau)) \le \left(\max_{0 \le s \le C_2.\tau^{-\frac{1}{\lambda}}\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}} |f(s)|\right)^{\frac{1}{p}}$.

Proof: The proof of this Theorem is most likely to the one of Theorem 3. Then, we leave it for the reader, who is interested in detail. Note that estimate (63) is just a combination of the a priori bound (39) and (25). □

Remark 26 *We emphasize that our existence results also holds for a class of continuous functions $f(u,x,t) : \mathbb{R} \times I \times (0,\infty) \to [0,\infty)$, such that $f(0,x,t) = 0$, $\forall (x,t) \in I \times (0,\infty)$; and for any $(x,t) \in I \times (0,\infty)$, $f(.,x,t)$ satisfies either (H_1), or (H_2), or a global Lipschitz property.*

4 Quenching phenomenon of nonnegative solutions

In this section, we will show that any weak solution of equation (1) must quench (Theorem 8 and Theorem 9). According to Proposition 24, it is enough to prove that the maximal solution u vanishes identically after a finite time. Then, we have the following result

Theorem 27 *Let $u_0 \in L^1(I)$, $u_0 \ge 0$, and f satisfy (H_2). Then, there exists a finite time T_0 such that*

$$u(x,t) = 0, \quad \forall x \in \overline{I}, \forall t > T_0.$$

Furthermore, T_0 can be estimated by a constant depending on $\beta, p, |I|, \|u_0\|_{L^1(I)}$.

Proof: For any $\tau > 0$, we put

$$L(\tau, u_0) = C(p, |I|).\tau^{-\frac{1}{\lambda}} \|u_0\|_{L^1(I)}^{\frac{p}{\lambda}},$$

the a priori bound of $u(x,t)$ on $[\tau, \infty)$, see (39).

Let $\Gamma_\varepsilon(t)$ be a solution of equation

$$\begin{cases} \partial_t \Gamma_\varepsilon(t) + g_\varepsilon(\Gamma_\varepsilon) = 0 \quad t > 0, \\ \Gamma_\varepsilon(0) = L(\tau, u_0). \end{cases} \tag{64}$$

Then, Γ_ε is a super-solution of equation (P_ε) satisfied by u_ε. Therefore, a comparison deduces

$$u_\varepsilon(x, s+\tau) \leq \Gamma_\varepsilon(s), \quad \forall (x,s) \in I \times (0, \infty).$$

It is straightforward to show that

$$\Gamma_\varepsilon(t) \overset{\varepsilon \to 0}{\longrightarrow} \Gamma(t) = \Big(L(\tau, u_0)^{1+\beta} - (1+\beta)t\Big)_+^{\frac{1}{1+\beta}}, \quad \text{for } t > 0.$$

Then, we obtain

$$u(x, s+\tau) \leq \Gamma(s), \quad \text{for } (x,s) \in I \times (0, \infty),$$

which implies

$$u(x,t) = 0, \quad \text{for any } t \geq \tau + \frac{1}{1+\beta} L^{1+\beta}(\tau, u_0), \text{ and for } x \in I. \tag{65}$$

Now, we try to estimate the value of the minimal extinction time T_0. It follows from (65) that

$$T_0 \leq \tau + \frac{1}{1+\beta} L^{1+\beta}(\tau, u_0), \quad \forall \tau > 0,$$

thereby

$$T_0 \leq \min_{\tau > 0} \{\tau + \frac{1}{1+\beta} L^{1+\beta}(\tau, u_0)\} = C(\beta, p, |I|) \|u_0\|_{L^1(I)}^{\frac{(1+\beta)p}{1+\beta+\lambda}}.$$

This completes the proof of Theorem 27, thereby proves Theorem 9. $\square$

Remark 28 *The result of Theorem 27 still holds if f is assumed as in Theorem 25.*

Remark 29 *Theorem 8 is proved similarly. Furthermore, T_0 can be estimated by the constant $\frac{\|u_0\|_\infty^{1+\beta}}{1+\beta}$, see also Theorem 35.*

As a consequence of Theorem 27, the existence result fails if $f(0) > 0$.

Corollary 30 *Let $f(u,x,t) : [0,\infty) \times I \times (0,\infty) \to [0,\infty)$ be a real nonnegative function. Assume that there is a point $x_0 \in I$ such that $f(0, x_0, t) > 0$, for any $t > 0$ large enough. Then, we have no nonnegative weak solution of problem (1).*

Proof: If the conclusion were false, there would exist then a weak solution of (1), say $\bar{u}$. Thus, $\bar{u}$ is a sub-solution of equation (34). Use the same analysis as in the proof of Theorem 27 to obtain

$$\bar{u}(x, s+\tau) \leq \Gamma(s), \quad \text{for } (x, s) \in I \times (0, \infty),$$

which implies that $\bar{u}(x,t)$ must vanish identically after a finite time T_0. In particular, we have from equation satisfied by $\bar{u}$ that $f(0, x_0, t) = 0$, for any $t > T_0$. This contradicts the above assumption, or we get Corollary 30. $\square$

Remark 31 *It is of course that our conclusion in Corollary 30 also holds for the inequalities* $u \geq 0$ *and*

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u, x, t) \leq 0 & \text{in } I \times (0, \infty), \\ u(-l, t) = u(l, t) = 0 & t \in (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } I. \end{cases}$$

5 On the associated Cauchy problem

In this section, we prove the existence results for the Cauchy problem (7). Furthermore, the behaviors of nonnegative solutions are considered such as the quenching phenomenon, the finite speed of propagation, and the ISS property.

5.1 Existence of a weak solution

As mentioned in the Introduction, we first have an existence result of problem (7).

Theorem 32 *Let* $0 \leq u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Assume that* f *satisfies either* (H_1), *or* (H_2), *or a global Lipschitz property. Then, there exists a weak bounded solution* $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R})) \cap L^p(0, T; W^{1,p}(\mathbb{R}))$, *satisfying equation* (7) *in* $\mathcal{D}'(\mathbb{R} \times (0,\infty))$. *Furthermore,* u *satisfies estimate* (10) *corresponding to* (H_1) *(resp. estimate* (26) *corresponding to* (H_2), *and estimate* (25) *corresponding to the assumption global Lipschitz).*

Proof: We only give the proof of the case (H_2). The case (H_1) (resp. global Lipschitz) is proved similarly.

Let u_r be the maximal solution of the following equation

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + u^{-\beta}\chi_{\{u>0\}} + f(u) = 0 & \text{in } I_r \times (0, \infty), \\ u(-r, t) = u(r, t) = 0, & \forall t \in (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } I_r, \end{cases} \tag{66}$$

see Theorem 3. It is clear that $\{u_r\}_{r>0}$ is a nondecreasing sequence. Moreover, the strong comparison principle deduces

$$u_r(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad \text{for } (x,t) \in I_r \times (0, \infty). \tag{67}$$

Thus, there exists a function u such that $u_r \uparrow u$ as $r \to \infty$. We will show that u is a solution of problem (7).

First, L^1-contraction provides us

$$\begin{cases} \|u_r(.,t)\|_{L^1(I_r)} \leq \|u_0\|_{L^1(\mathbb{R})}, \quad \text{for any } t \in (0,\infty), \\ \\ \|f(u_r)\|_{L^1(I_r\times(0,\infty))}, \ \|u_r^{-\beta}\chi_{\{u_r>0\}}\|_{L^1(I_r\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R})}. \end{cases} \tag{68}$$

It follows immediately from the Monotone Convergence Theorem that $u_r(t)$ converges to $u(t)$ in $L^1(\mathbb{R})$ and $f(u_r)$ converges to $f(u)$ in $L^1(\mathbb{R}\times(0,\infty))$ as $r\to\infty$, likewise

$$\begin{cases} \|u(.,t)\|_{L^1(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}, \quad \text{for any } t \in (0,\infty), \\ \\ \|f(u)\|_{L^1(\mathbb{R}\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R})}. \end{cases} \tag{69}$$

Next, for any $r>0$, we have from (10) (see also Theorem 19)

$$|\partial_x u_r(x,t)| \leq C.u_r^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{1+\beta}{p}} + \|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\beta}{p}}.f^{\frac{1}{p}}\left(2\|u_0\|_{L^\infty(\mathbb{R})}\right)+1\right), \tag{70}$$

for a.e $(x,t)\in I_r\times(0,\infty)$. By using the same argument as in the proof of (48), there is a subsequence of $\{u_r\}_{r>0}$ such that $\partial_x u_r \overset{r\to\infty}{\longrightarrow} u_x$, for a.e $(x,t)\in\mathbb{R}\times(0,\infty)$. From this result and (70), we obtain

$$|u_x(x,t)| \leq C.u^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}\|u_0\|_{L^\infty(\mathbb{R})}^{\frac{1+\beta}{p}} + \|u_0\|_{L^\infty(\mathbb{R})}^{\frac{\beta}{p}}.f^{\frac{1}{p}}\left(2\|u_0\|_{L^\infty(\mathbb{R})}\right)+1\right), \tag{71}$$

for a.e $(x,t)\in\mathbb{R}\times(0,\infty)$, and

$$\partial_x u_r \overset{r\to\infty}{\longrightarrow} u_x, \quad \text{in } L^q_{loc}(\mathbb{R}\times(0,\infty)), \quad \forall q\geq 1. \tag{72}$$

Now, we show that u satisfies equation (7) in the sense of distribution. Indeed, using the test function $\psi_\eta(u_r).\phi$ for the equation satisfied by u_r gives us

$$\int_{Supp(\phi)} -\Psi_\eta(u_r)\phi_t + |\partial_x u_r|^{p-2}\partial_x u_r.\phi_x\psi_\eta(u_r) + \frac{1}{\eta}|\partial_x u_r|^{p-2}\partial_x u_r.\psi'(\frac{u_r}{\eta})\phi$$
$$+ u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi + f(u_r)\psi_\eta(u_r)\phi \, dsdx = 0, \quad \forall\phi\in\mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty)).$$

We first take care of the term $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r)\phi$ in passing $r\to\infty$ and $\eta\to 0$. It is not difficult to see that $u_r^{-\beta}\chi_{\{u_r>0\}}\psi_\eta(u_r) = u_r^{-\beta}\psi_\eta(u_r)$ is bounded by $\eta^{-\beta}$. Then for any $\eta>0$, the Dominated Convergence Theorem yields $u_r^{-\beta}\psi_\eta(u_r) \overset{r\to\infty}{\longrightarrow} u^{-\beta}\psi_\eta(u)$ in $L^1_{loc}(\mathbb{R}\times(0,\infty))$, which implies

$$\|u^{-\beta}\psi_\eta(u)\|_{L^1(\mathbb{R}\times(0,\infty))} \overset{(68)}{\leq} \|u_0\|_{L^1(\mathbb{R})}.$$

Next, using the Monotone Convergence Theorem deduces $u^{-\beta}\psi_\eta(u)\uparrow u^{-\beta}\chi_{\{u>0\}}$ as $\eta\to 0$, thereby proves

$$\|u^{-\beta}\chi_{\{u>0\}}\|_{L^1(\mathbb{R}\times(0,\infty))} \leq \|u_0\|_{L^1(\mathbb{R})}. \tag{73}$$

Thanks to (72), (68) and (67), there is no problem of passing to the limit as $r\to\infty$ in the indicated variational equation in order to get

$$\int_{Supp(\phi)} -\Psi_\eta(u)\phi_t + |u_x|^{p-2}u_x.\phi_x\psi_\eta(u) + \frac{1}{\eta}|u_x|^{p-2}u_x.\psi'(\frac{u}{\eta})\phi$$
$$+ u^{-\beta}\psi_\eta(u)\phi + f(u)\psi_\eta(u)\phi \, dsdx = 0, \quad \forall\phi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty)).$$

By (69), (71), and (73), we can make the same argument as in (58) and (59) to obtain after letting $\eta \to 0$

$$\int_{Supp(\phi)} -u\phi_t + |u_x|^{p-2}u_x.\phi_x + u^{-\beta}\chi_{\{u>0\}}\phi + f(u)\phi \, dxds = 0, \quad \forall\phi \in \mathcal{C}_c^\infty(\mathbb{R}\times(0,\infty)). \quad (74)$$

Or u satisfies equation (1) in the sense of distribution.

Then, it remains to prove that $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$. Let us first claim that

$$u \in \mathcal{C}([0,\infty); L^1_{loc}(\mathbb{R})). \quad (75)$$

In order to prove (75), we borrow a compactness result of A. Porretta [25]. We present it here for a convenience

Lemma 33 (Theorem 1.1, **[25])** *Let* $p > 1$ *and* p' *its conjugate exponent* $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$, $a, b \in \mathbb{R}$*, and define the space*

$$V_1^p(a,b) = \{u : \Omega\times(a,b) \to \mathbb{R}; \quad u \in L^p(a,b; W_0^{1,p}(\Omega)),$$
$$u_t \in L^{p'}(a,b; W^{-1,p'}(\Omega)) + L^1(\Omega\times(a,b))\},$$

where Ω *is a bounded set in* $\mathbb{R}^N$*. Then, we have*

$$V_1^p(a,b) \subset \mathcal{C}([a,b]; L^1(\Omega)).$$

Proof: See its proof in Theorem 1.1, [25]. □

For any $r > 0$, we extend u_r by 0 outside I_r, still denoted as u_r. Use u_r as a test function to the equation satisfied by u_r to get

$$\int_0^T \int_{\mathbb{R}} |\partial_x u_r|^p dxds \le \frac{1}{2}\int_{I_r} u_0^2(x)dx \le \frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}\|u_0\|_{L^\infty(\mathbb{R})}, \quad \text{for } T > 0.$$

Thus $\|u_x\|^p_{L^p(\mathbb{R}\times(0,T))}$ is also bounded by $\frac{1}{2}\|u_0\|_{L^1(\mathbb{R})}\|u_0\|_{L^\infty(\mathbb{R})}$. By (69) and the boundedness of u, it follows from the Interpolation Theorem that $u \in L^p(\mathbb{R}\times(0,T))$, for any $T > 0$. Thus, we have $u \in L^p(0,T; W^{1,p}(\mathbb{R}))$.
According to this conclusion, (69) and (73), we have from the equation of u

$$u_t \in L^{p'}(a,b; W^{-1,p'}(\mathbb{R})) + L^1(\mathbb{R}\times(0,T)).$$

Then, a local argument of Lemma 33 yields the claim (75). (Note that the last conclusion does not implies $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$ since the proof of Theorem 1.1, [25] depends on the boundedness of Ω. Moreover, the proof of (57) is not applicable to prove (75), since the solution u of the

Cauchy problem is constructed in a different way)

Now, to prove $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$, it suffices to show that $u(t)$ is continuous at $t = 0$ in $L^1(\mathbb{R})$, i.e

$$\lim_{t\to 0}\int_{\mathbb{R}} |u(x,t) - u_0(x)|dx = 0,$$

and the conclusion for $t > 0$ is proved in the same way. In fact, we have for any $m \geq 1$

$$\int_{\mathbb{R}} |u(x,t) - u_0(x)|dx \leq \int_{I_m} |u(x,t) - u_0(x)|dx + \int_{\mathbb{R}\setminus I_m} |u(x,t) - u_0(x)|dx$$

$$\leq \int_{I_m} |u(x,t) - u_0(x)|dx + \int_{\mathbb{R}\setminus I_m} u(x,t)dx + \int_{\mathbb{R}\setminus I_m} u_0(x)dx =$$

$$\int_{I_m} |u(x,t) - u_0(x)|dx - \left(\int_{I_m} (u(x,t) - u_0(x))dx\right) + \int_{\mathbb{R}} u(x,t)dx - \int_{I_m} u_0(x)dx + \int_{\mathbb{R}\setminus I_m} u_0(x)dx$$

$$\overset{(69)}{\leq} 2\int_{I_m} |u(x,t) - u_0(x)|dx + \int_{\mathbb{R}} u_0(x)dx - \int_{I_m} u_0(x)dx + \int_{\mathbb{R}\setminus I_m} u_0(x)dx =$$

$$2\int_{I_m} |u(x,t) - u_0(x)|dx + 2\int_{\mathbb{R}\setminus I_m} u_0(x)dx.$$

Taking $\limsup_{t\to 0}$ both sides of the indicated inequality deduces

$$\limsup_{t\to 0}\int_{\mathbb{R}} |u(x,t) - u_0(x)|dx \leq 2\limsup_{t\to 0}\int_{I_m} |u(x,t) - u_0(x)|dx + 2\int_{\mathbb{R}\setminus I_m} u_0(x)dx.$$

By $u \in \mathcal{C}([0,\infty); L^1_{loc}(\mathbb{R}))$, we obtain from the last inequality

$$\limsup_{t\to 0}\int_{\mathbb{R}} |u(x,t) - u_0(x)|dx \leq 2\int_{\mathbb{R}\setminus I_m} u_0(x)dx.$$

Then the result follows as $m \to \infty$. Or, we complete the proof of Theorem 32. $\square$

Remark 34 *It is obvious that $u \in \mathcal{C}([0,\infty); L^q(\mathbb{R}))$, for any $q \geq 1$. Thus, $u(0) = u_0$ in $L^q(\mathbb{R})$.*

Next, we show that the quenching phenomenon still holds for any weak nonnegative solutions of the Cauchy problem (7).

Theorem 35 *Let v be such a solution of problem (7). Then, v must vanish identically after a finite time $T_0 > 0$. Moreover, T_0 can be estimated by $\frac{\|u_0\|^{1+\beta}_{L^\infty(\mathbb{R})}}{1+\beta}$.*

Proof: It is not difficult to observe that

$$\begin{cases} \partial_t v - (|v_x|^{p-2}v_x)_x + g_\varepsilon(v) + f(v)\psi_\varepsilon(v) \leq 0, & \text{in } \mathbb{R}\times(0,\infty), \\ v(x,0) = u_0(x), & \text{in } \mathbb{R}. \end{cases} \tag{76}$$

Remind that $g_\varepsilon(.)$ is a global Lipschitz function, while $f(.)\psi_\varepsilon(.)$ is a non-decreasing function. These facts allow us to apply the strong comparison principle in order to get

$$v(x,t) \leq \Gamma_\varepsilon(t), \quad \forall(x,t) \in \mathbb{R}\times(0,\infty),$$

where Γ_ε is in (64) with initial data $\|u_0\|_{L^\infty(\mathbb{R})}$.
Letting $\varepsilon \to 0$ deduces

$$v(x,t) \leq \Gamma(t) = \left(\|u_0\|^{1+\beta}_{L^\infty(\mathbb{R})} - (1+\beta)t\right)_+^{\frac{1}{1+\beta}}, \quad \forall (x,t) \in \mathbb{R} \times (0,\infty).$$

This completes the proof of Theorem 35. □

5.2 Existence of a maximal solution with compact support initially

In general, we have no answer for the existence of a maximal solution of the Cauchy problem. However, we will show that the solution u, constructed in Theorem 32 is a maximal solution if the initial data has compact support.

Theorem 36 *Assume that Supp$(u_0) \subset I_{R_0}$. Then, the solution u constructed as in Theorem 32 is a maximal solution of equation (7). Moreover, Supp$\,(u(t))$ is bounded for all $t > 0$.*

Proof: First, we have the following Lemma, which refers to the finite speed of propagation of nonnegative solutions.

Lemma 37 *Let v be a weak solution of equation (7). Then, v has compact support at all later time $t > 0$. Moreover, we have*

$$\textit{Supp}\,(v(t)) \subset I_{m_0}, \quad \textit{for any } t > 0,$$

with $m_0 = R_0 + \dfrac{\|u_0\|^{\frac{1}{\gamma}}_{L^\infty(\mathbb{R})}}{\sigma}$, *and* $\sigma = \left(\dfrac{1}{\gamma^{p-1}(\gamma-1)(p-1)}\right)^{\frac{1}{p}}$.

Proof: For any $\varepsilon > 0$, let w_ε be a nonnegative solution of the following equation

$$\begin{cases} -\partial_x\left(|\partial_x w_\varepsilon|^{p-2}\partial_x w_\varepsilon\right) + g_\varepsilon(w_\varepsilon) = 0, & \text{in } \mathbb{R}^+, \\ w_\varepsilon(0) = \|u_0\|_{L^\infty(\mathbb{R})}, \\ \lim\limits_{x\to\infty} w_\varepsilon(x) = 0. \end{cases} \tag{77}$$

It is straight forward that

$$w_\varepsilon(x) \overset{\varepsilon\to 0}{\longrightarrow} w(x) = \left(\|u_0\|^{\frac{1}{\gamma}}_{L^\infty} - \sigma.x\right)_+^{\gamma}, \quad \text{for } x > 0.$$

If we can show that

$$v(x,t) \leq w(x - R_0), \quad \text{for } x > R_0,\ t > 0, \tag{78}$$

then $v(x,t) = 0$, for any $x \geq m_0$, and for $t > 0$. The same argument for the case $x < -R_0$ implies $v(x,t) = 0$, for any $x \leq -m_0$, and for $t > 0$, thereby proves the above Lemma.

Now, we prove (78). Recall that v satisfies (76) in $(R_0,\infty)\times(0,\infty)$. Moreover, we have

$$\begin{cases} v(x,t)\mid_{x=R_0} \leq \|u_0\|_{L^\infty} = w_\varepsilon(x - R_0)\mid_{x=R_0}, \\ \\ v(x,0) = 0 \leq w_\varepsilon(x - R_0), \quad \text{for } x > R_0. \end{cases}$$

By comparison principle, we obtain

$$v(x,t) \leq w_\varepsilon(x), \quad \text{for } (x,t) \in (R_0,\infty) \times (0,\infty).$$

Letting $\varepsilon \to 0$ yields conclusion (78). This puts an end to the proof of Lemma 37. $\square$

It suffices to prove that u is a maximal solution of problem (7). Indeed, let v be a weak solution of problem (7). Thanks to Lemma 37, $v\mid_{I_{m_0}\times(0,\infty)}$ is a weak solution of the following problem

$$\begin{cases} \partial_t v - (|v_x|^{p-2}v_x)_x + v^{-\beta}\chi_{\{v>0\}} + f(v) = 0 & \text{in } I_{m_0} \times (0,\infty), \\ v(-m_0,t) = v(m_0,t) = 0 & t \in (0,\infty), \\ v(x,0) = u_0(x) & \text{in } I_{m_0}. \end{cases}$$

This implies that $v(x,t) \leq u_r(x,t)$, in $\mathbb{R} \times (0,\infty)$, for any $r \geq m_0$. Passing $r \to \infty$ completes the proof of Theorem 36. $\square$

Remark 38 *Thanks to Lemma 37, we observe that $u_r = u$, in $\mathbb{R} \times (0,\infty)$, for any $r \geq m_0$. Thus, considering the Cauchy problem* (7) *is equivalent to considering Dirichlet problem* (1) *in the case of compact support initially.*

5.3 Instantaneous shrinking of compact support

In this section, we will show that if f satisfies a certain growth condition at infinity, see (H_3), the ISS phenomenon occurs then for any nonnegative solution of equation (7). It is of course that there are many functions satisfying either (H_1) and (H_3), or (H_2) and (H_3). We can take for example: $f(s) = s^q$, for some $q > 0$; or $f(s) = e^s - 1$; and so forth. After that, we have the following result

Theorem 39 *Let f satisfy either (H_1) and (H_3), or (H_2) and (H_3). Assume that $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $u_0(x)$ tends to 0 uniformly as $|x| \to \infty$. Then any nonnegative solution of equation* (7) *has ISS property.*

Proof: Let v be a solution of equation (7). From (H_3), there is a real number $R_0 > 0$ large enough such that $f(s) \geq s^{q_0}$, for $s \geq R_0$. Thus, we have

$$v^{-\beta}\chi_{\{v>0\}} + f(v) \geq R_0^{-(\beta+q_0)}.v^{q_0},$$

which leads to

$$\partial_t v - (|v_x|^{p-2}v_x)_x + R_0^{-(\beta+q_0)}.v^{q_0} \leq 0, \quad \text{in } \mathbb{R} \times (0,\infty).$$

Let y be a unique solution of the following problem

$$\begin{cases} \partial_t y - (|y_x|^{p-2}y_x)_x + R_0^{-(\beta+q_0)}.y^{q_0} = 0, & \text{in } \mathbb{R} \times (0,\infty), \\ y(x,0) = u_0(x), & \text{in } \mathbb{R}, \end{cases}$$

(see e.g, [16], [29], [30], and [12]). By the strong comparison principle, we get

$$v(x,t) \leq y(x,t), \quad \text{in } \mathbb{R} \times (0,\infty).$$

Moreover, y has the ISS property, see [16], so does v. This puts an end to the proof of the above Theorem. $\square$

Remark 40 *We also note that the result of Theorem* 35*, Theorem* 36*, and Theorem* 39 *still hold for the case where f is a global Lipschitz function, and $f(0) = 0$.*

6 Appendix

Proof of Lemma 22: A subtraction between two equations satisfied by v_1 and v_2 gives us

$$\partial_t(v_1 - v_2) - \partial_x \left(|\partial_x v_1|^{p-2}\partial_x v_1 - |\partial_x v_2|^{p-2}\partial_x v_2\right) + g_\varepsilon(v_1) - g_\varepsilon(v_2) + f\psi_\varepsilon(v_1) - f\psi_\varepsilon(v_2) \leq 0.$$

Multiplying both sides of the above equation with the test function $T_1(w)$, $w = (v_1 - v_2)_+$; and using integration by parts yield

$$\int_I S_1(w(x,t))dx + \int_\tau^t \int_I \left(|\partial_x v_1|^{p-2}\partial_x v_1 - |\partial_x v_2|^{p-2}\partial_x v_2\right)(\partial_x T_1(w))\, dxds+$$
$$\int_\tau^t \int_I (g_\varepsilon(v_1) - g_\varepsilon(v_2)) .T_1(w)dxds + \int_\tau^t \int_I (f\psi_\varepsilon(v_1) - f\psi_\varepsilon(v_2)) .T_1(w)dxds \leq \int_I S_1(w(x,\tau))dx,$$

for $t > \tau > 0$. It follows from the monotone of p-Laplacian, and the monotone of $f\psi_\varepsilon$, and the fact that g_ε is a global Lipschitz function

$$\int_I S_1(w(x,t))dx \leq C(\varepsilon)\int_\tau^t \int_I |v_1 - v_2|T_1(w)dxds + \int_I S_1(w(x,\tau))dx,$$

where $C(\varepsilon) > 0$ is the Lipschitz constant of g_ε. Letting $\tau \to 0$ in the above inequality deduces

$$\int_I S_1(w(x,t))dx \leq C(\varepsilon)\int_0^t \int_I |v_1 - v_2|T_1(w)dxds.$$

In addition, we have

$$|v_1 - v_2|T_1(w)(x,t) \leq 2S_1(w(x,t)).$$

Inserting this fact into the indicated inequality yields

$$\int_I S_1(w(x,t))dx \leq 2C(\varepsilon)\int_0^t \int_I S_1(w(x,t))dxds.$$

Then, we arrive to the following ordinary differential equation

$$\begin{cases} \frac{d}{dt}y(t) \leq 2C(\varepsilon)y(t), & t > 0, \\ y(0) = 0. \end{cases}$$

with

$$y(t) = \int_I S_1(w(x,t))dx.$$

It follows from Gronwall's lemma that

$$y(t) = 0, \quad \forall t > 0,$$

which implies

$$w(t) = 0, \quad \forall t > 0.$$

In other words, we get the above lemma.

Remark 41 *The result of Lemma 22 also holds for any sub-solution v_1 and super-solution v_2 of equation* (34) *satisfying $v_2 \geq v_1$ on the boundary.*

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